Supplement to "Historical Reflections on Teaching Trigonometry": Ptolemy

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Claudius Ptolemy of Alexandria (*circa* 85–165 CE) established the basis for Western, South Asian, and Middle Eastern astronomy that would last until the 16th century. Ptolemy's book, known originally as "The Mathematical Treatise," would come to be known as "The Great Treatise" or, in Arabic, *Kitāb al-majisī*, which was translated into Latin as the *Almagest*. One of the basic challenges that Ptolemy had to face was constructing a table of chords corresponding to various arc lengths.

First he had to choose the radius of his circle, recognizing that chord lengths would need to be scaled when applied to specific circles. He chose a radius of $R = 60$. Since, following the practice of the ancient Mesopotamians, each degree was subdivided into 60 minutes and each minute into 60 seconds and so on, the choice of 60 simply makes it easy to scale, much as choosing a radius of 100 would be convenient for our decimal system. We will explain his results in terms of an arbitrary radius, R.

As explained in the print article and the supplement on Euclid, Ptolemy started with a knowledge of the chords (crd) of several arc lengths, especially crd $60^\circ = R$ and crd $72^\circ = \sqrt{(5-\sqrt{5})/2}\ R$. The next step in building his table was to show how to find the chord for an arc length that is the sum or difference of arc lengths for which the chords are known. This would enable him to find the chord length for 12°, and then successively cut that in half to get down to the chord of $3/4^{\text{ths}}$ of a degree, not quite what he needed, but getting close. The key to the sum and difference of arc lengths formula is what we now know as Ptolemy's Theorem.

Ptolemy's Theorem. *Given any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.*

Proof. The proof relies on Euclid's result, Book III, Proposition 21, that if we take any chord $\overline{AB}$ of a circle and any third point C on the circle, then the angle $\angle ACB$ depends only on the chord $\overline{AB}$ and not on the choice of C. In fact, $\angle ACB$ is exactly half the length of the arc from A to B, a result that we shall need later. It follows that in Figure 1,

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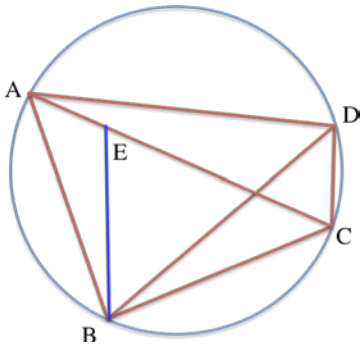

Figure 1: For any quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides: $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

$\angle BAC = \angle BDC$. We draw a line segment from B, meeting $\overline{AC}$ at E, so that $\angle ABE = \angle DBC$. It follows that triangles ABE and DBC are similar, and, therefore

$$\frac{AE}{AB} = \frac{CD}{BD} \quad \Longrightarrow \quad BD \cdot AE = AB \cdot CD.$$

Again invoking Book III, Proposition 21, we see that $\angle ADB = \angle ACB$. From the construction of $\overline{BE}$, we also have that $\angle ABD = \angle CBE$. Now it follows that triangles ADB and ECB are similar, and, therefore

$$\frac{BD}{AD} = \frac{BC}{EC} \quad \Longrightarrow \quad BD \cdot EC = AD \cdot BC.$$

Combining these results, we obtain

$$BD \cdot AC = BD \cdot (AE + EC) = BD \cdot AE + BD \cdot EC = AB \cdot CD + AD \cdot BC. \qquad (1)$$

□

To get the sum and difference of angles formulas, we consider the special case of this theorem in which one of the diagonals of the quadrilateral is a diameter of the circle of radius R (see Figure 2 in which α is the arc length from A to D and β is the arc length from A to B. The diameter is $AC = 2R$). Note that for any arc length α,

$$(\text{crd } \alpha)^2 + \big(\text{crd } (180^\circ - \alpha)\big)^2 = (2R)^2.$$

If we know the lengths of chords $\overline{AB}$ and $\overline{AD}$, then we know the lengths of all chords except $\overline{BD}$, and Ptolemy's Theorem can be used to find this chord length (crd):

$$\text{crd } (\alpha) \cdot \text{crd } (180^\circ - \beta) + \text{crd } (\beta) \cdot \text{crd } (180^\circ - \alpha) = 2R \,\text{crd } (\alpha + \beta). \qquad (2)$$

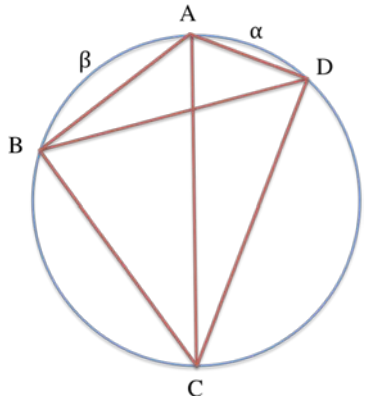

Figure 2: In terms of chords (crd), the sum of angles formula for the sine translates as $\text{crd}\ (\alpha) \cdot \text{crd}\ (180^\circ - \beta) + \text{crd}\ (\beta) \cdot \text{crd}\ (180^\circ - \alpha) = 2R\,\text{crd}\ (\alpha + \beta)$, a direct consequence of Ptolemy's Theorem.

Similarly, if we know the chord of α and the chord of $\alpha + \beta$, Ptolemy's Theorem enables us to find the chord of β.

Ptolemy could have derived the half angle formula by setting $\alpha = \beta$ in equation (2) and solving for crd α in terms of crd 2α. He chose instead to derive the formula

$$(\text{crd}\ \alpha)^2 = 2R^2 - R\,\text{crd}\ (180^\circ - 2\alpha) \tag{3}$$

as follows (see Figure 3). We draw a diameter $\overline{AB}$ and place points C and D so that the arc length from B to C and the arc length from C to D both equal α. We drop a perpendicular from C to $\overline{AB}$, meeting $\overline{AB}$ at F, and locate the point E so that $AD = AE$. Since $\angle DAC = \angle CAE$, triangles ADC and AEC are congruent. Since $CE = CD = BC$, triangles CFE and CFB are congruent, implying that

$$BF = \frac{1}{2}BE = \frac{1}{2}(AB - AE) = \frac{1}{2}(AB - AD). \tag{4}$$

Triangles ACB and CFB are similar, so

$$\frac{BC}{BF} = \frac{AB}{BC} \quad \Longrightarrow \quad BC^2 = AB \cdot BF = \frac{AB}{2}(AB - AD). \tag{5}$$

Using the fact that $AB = 2R$, $BC = \text{crd}\ \alpha$, and $AD = \text{crd}\ (180^\circ - 2\alpha)$, we get equation (3).

The results obtained so far enabled Ptolemy to find the exact value of the chord of any arc of length $3 \cdot 2^k$ degrees where k can be any integer. In the 12th century, this fact would lead Al-Samawal to argue that the circle should be divided into 480° rather than 360°,

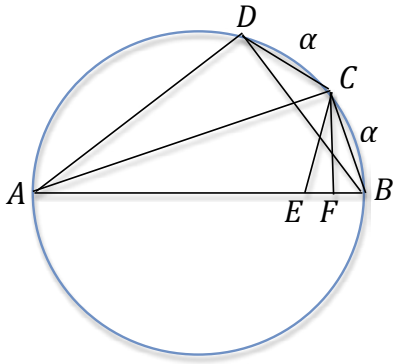

Figure 3: In terms of chords, the half angle formula can be expressed as $(\text{crd}\ \alpha)^2 = 2R^2 - R\,\text{crd}\ (180^\circ - 2\alpha)$.

because it *is* possible to find an exact value for the chord of $1/480^{\text{th}}$ of the circumference of a circle. He seems to have convinced no one to change the definition of a degree.

Although Ptolemy could not find an exact value for the chord of 1°, he was able to create a table of chord lengths for all arc lengths from 0° to 90° in increments of half a degree and to within an accuracy of one part in $216{,}000 = 60^3$. The following proposition enabled him to calculate crd 1° and crd $30'$ to the desired accuracy.

Proposition. *If* $0 < \alpha < \beta < 180^\circ$ *are arc lengths, then*

$$\frac{\text{crd}\ \beta}{\text{crd}\ \alpha} < \frac{\beta}{\alpha}. \tag{6}$$

In particular, this implies that

$$\frac{2}{3}\,\text{crd}\ 1^\circ 30' < \text{crd}\ 1^\circ < \frac{4}{3}\,\text{crd}\ 45',$$

bounds that produce the desired accuracy. In fact, these bounds differ by less than one part in 2,600,000.

Proof. See Figures 4 and 5. Let α denote the arc length from A to B and β the arc length from B to C. Draw the line segment $\overline{BD}$ that bisects $\angle ABC$, and mark E as the point of intersection of $\overline{AC}$ and $\overline{BD}$.

We will need the fact that, since $\angle ABE = \angle EBC$,

$$\frac{AB}{AE} = \frac{BC}{CE}. \tag{7}$$

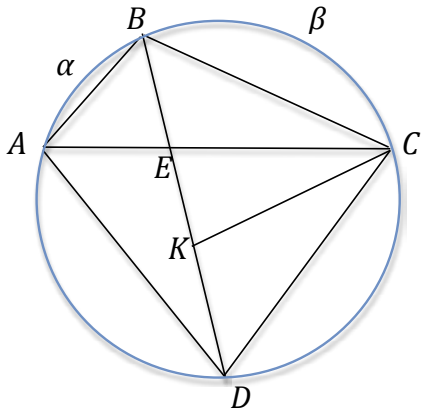

Figure 4: If ABC is any triangle and $\overline{BE}$ bisects the interior angle at B, then the ratio of AB to AE is equal to the ratio of BC to CE: $\dfrac{AB}{AE} = \dfrac{BC}{CE}$.

We can see why this is true if we add a point K on the segment $\overline{BD}$ so that $CE = CK$. Since triangle ECK is isosceles, $\angle EKC = \angle KEC = \angle AEB$, and therefore triangles ABE and CBK are similar. From this it follows that

$$\frac{AB}{AE} = \frac{BC}{CK} = \frac{BC}{CE}.$$

We draw a perpendicular from D to $\overline{AC}$, meeting $\overline{AC}$ at F (see Figure 5). Since the arc from A to D is equal to the arc from D to C, F is at the midpoint of $\overline{AC}$. We draw the arc of the circle centered at D with radius DE and mark its intersection with $\overline{AD}$ as G and its intersection with the extension of $\overline{DF}$ as H.

Since triangles EFD and AED have the same heights, the ratio of their areas is equal to the ratio of their bases, EF/AE. The ratio of the areas of these triangles is less than the ratio of the areas of the sectors EHD to GED, and the ratio of the sectors is equal to the ratio of the angles at D. We see that

$$\frac{EF}{AE} = \frac{\text{area of } \Delta EFD}{\text{area of } \Delta AED} < \frac{\text{area of sector } EHD}{\text{area of sector } GED} = \frac{\angle EDF}{\angle ADE}. \tag{8}$$

We observe that

$$EC = EF + FC = EF + AF = 2EF + AE,$$

and, similarly, $\angle EDC = 2\angle EDF + \angle ADE$. Now we put it all together,

$$\begin{aligned} \frac{\text{crd } \beta}{\text{crd } \alpha} = \frac{BC}{AB} = \frac{EC}{AE} &= 2\frac{EF}{AE} + \frac{AE}{AE} \\ &< 2\frac{\angle EDF}{\angle ADE} + \frac{\angle ADE}{\angle ADE} = \frac{\angle EDC}{\angle ADE} = \frac{\beta/2}{\alpha/2} = \frac{\beta}{\alpha}. \end{aligned} \tag{9}$$

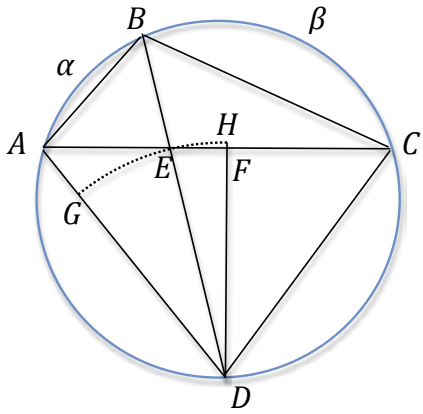

Figure 5: For ratios larger than 1, the ratio of the chord lengths is strictly less than the ratio of the corresponding arc lengths: $\dfrac{\text{crd } \beta}{\text{crd } \alpha} < \dfrac{\beta}{\alpha}$.

□

Now that Ptolemy knew the chord of $\frac{1}{2}^\circ = 30'$ and had exact values at each multiple of $1^\circ\,30'$, he could find very accurate values for chords at any multiple of half a degree. Of course, as we saw in the problem of locating the position of the earth, we need finer values than this. In his table, Ptolemy reported the value of one sixtieth of the difference between each pair of successive chord values. With this information, anyone using his table could employ linear interpolation to find the intermediate chord values for arc lengths in increments of half a minute.